Lecture 1

Introduction, Maxwell's Equations

1.1 Importance of Electromagnetics

We will explain why electromagnetics is so important, and its impact on very many different areas. Then we will give a brief history of electromagnetics, and how it has evolved in the modern world. Then we will go briefly over Maxwell's equations in their full glory. But we will begin the study of electromagnetics by focussing on static problems.

The discipline of electromagnetic field theory and its pertinent technologies is also known as electromagnetics. It has been based on Maxwell's equations, which are the result of the seminal work of James Clerk Maxwell completed in 1865, after his presentation to the British Royal Society in 1864. It has been over 150 years ago now, and this is a long time compared to the leaps and bounds progress we have made in technological advancements. But despite, research in electromagnetics has continued unabated despite its age. The reason is that electromagnetics is extremely useful, and has impacted a large sector of modern technologies.

To understand why electromagnetics is so useful, we have to understand a few points about Maxwell's equations.

- First, Maxwell's equations are valid over a vast length scale from subatomic dimensions to galactic dimensions. Hence, these equations are valid over a vast range of wavelengths, going from static to ultra-violet wavelengths.[1]

- Maxwell's equations are relativistic invariant in the parlance of special relativity [1]. In fact, Einstein was motivated with the theory of special relativity in 1905 by Maxwell's equations [2]. These equations look the same, irrespective of what inertial reference frame one is in.

- Maxwell's equations are valid in the quantum regime, as it was demonstrated by Paul Dirac in 1927 [3]. Hence, many methods of calculating the response of a medium to

[1]Current lithography process is working with using ultra-violet light with a wavelength of 193 nm.

classical field can be applied in the quantum regime also. When electromagnetic theory is combined with quantum theory, the field of quantum optics came about. Roy Glauber won a Nobel prize in 2005 because of his work in this area [4].

- Maxwell's equations and the pertinent gauge theory has inspired Yang-Mills theory (1954) [5], which is also known as a generalized electromagnetic theory. Yang-Mills theory is motivated by differential forms in differential geometry [6]. To quote from Misner, Thorne, and Wheeler, "Differential forms illuminate electromagnetic theory, and electromagnetic theory illuminates differential forms." [7, 8]
- Maxwell's equations are some of the most accurate physical equations that have been validated by experiments. In 1985, Richard Feynman wrote that electromagnetic theory has been validated to one part in a billion.[2] Now, it has been validated to one part in a trillion (Aoyama et al, Styer, 2012).[3]
- As a consequence, electromagnetics has had a tremendous impact in science and technology. This is manifested in electrical engineering, optics, wireless and optical communications, computers, remote sensing, bio-medical engineering etc.

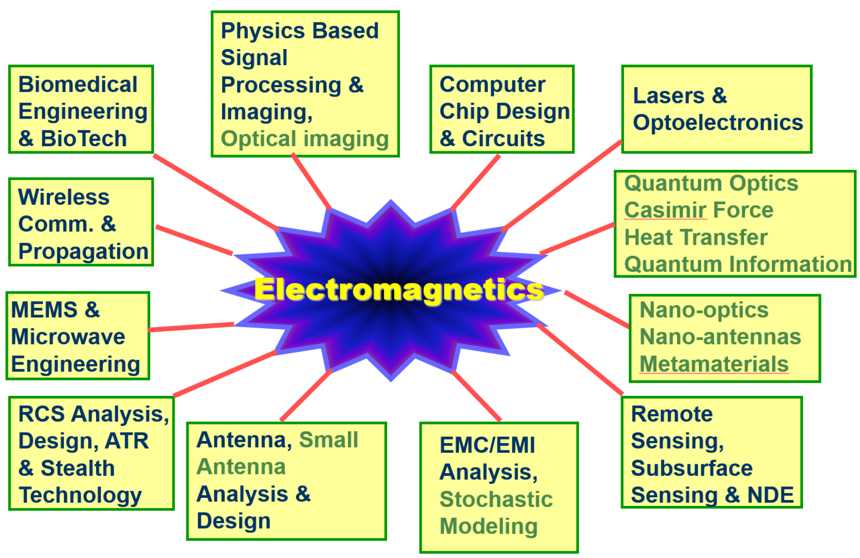

Figure 1.1: The impact of electromagnetics in many technologies. The areas in blue are prevalent areas impacted by electromagnetics some 20 years ago [9], and the areas in red are modern emerging areas impacted by electromagnetics.

[2]This means that if a jet is to fly from New York to Los Angeles, an error of one part in a billion means an error of a few millmeters.

[3]This means an error of a hairline, if one were to fly from the earth to the moon.

1.2 A Brief History of Electromagnetics

Electricity and magnetism have been known to humans for a long time. Also, the physical properties of light has been known. But electricity and magnetism, now termed electromagnetics in the modern world, has been thought to be governed by different physical laws as opposed to optics. This is understandable as the physics of electricity and magnetism is quite different of the physics of optics as they were known to humans.

For example, lode stone was known to the ancient Greek and Chinese around 600 BC to 400 BC. Compass was used in China since 200 BC. Static electricity was reported by the Greek as early as 400 BC. But these curiosities did not make an impact until the age of telegraphy. The coming about of telegraphy was due to the invention of the voltaic cell or the galvanic cell in the late 1700's, by Luigi Galvani and Alesandro Volta [10]. It was soon discovered that two pieces of wire, connected to a voltaic cell, can be used to transmit information.

So by the early 1800's this possibility had spurred the development of telegraphy. Both Andr-Marie Ampre (1823) [11, 12] and Michael Faraday (1838) [13] did experiments to better understand the properties of electricity and magnetism. And hence, Ampere's law and Faraday law are named after them. Kirchhoff voltage and current laws were also developed in 1845 to help better understand telegraphy [14, 15]. Despite these laws, the technology of telegraphy was poorly understood. It was not known as to why the telegraphy signal was distorted. Ideally, the signal should be a digital signal switching between one's and zero's, but the digital signal lost its shape rapidly along a telegraphy line.[4]

It was not until 1865 that James Clerk Maxwell [17] put in the missing term in Ampere's law, the term that involves displacement current, only then the mathematical theory for electricity and magnetism was complete. Ampere's law is now known as generalized Ampere's law. The complete set of equations are now named Maxwell's equations in honor of James Clerk Maxwell.

The rousing success of Maxwell's theory was that it predicted wave phenomena, as they have been observed along telegraphy lines. Heinrich Hertz in 1888 [18] did experiment to proof that electromagnetic field can propagate through space across a room. Moreover, from experimental measurement of the permittivity and permeability of matter, it was decided that electromagnetic wave moves at a tremendous speed. But the velocity of light has been known for a long while from astronomical observations (Roemer, 1676) [19]. The observation of interference phenomena in light has been known as well. When these pieces of information were pieced together, it was decided that electricity and magnetism, and optics, are actually governed by the same physical law or Maxwell's equations. And optics and electromagnetics are unified into one field.

[4] As a side note, in 1837, Morse invented the Morse code for telegraphy [16]. There were cross pollination of ideas across the Atlantic ocean despite the distance. In fact, Benjamin Franklin associated lightning with electricity in the latter part of the 18-th century. Also, notice that electrical machinery was invented in 1832 even though electromagnetic theory was not fully understood.

- Lode stone 400BC, Compass 200BC
- Static electricity, Greek, 400 BC
- Ampere's Law 1823;
- Faraday Law 1838;
- KCL/KVL 1845
- Telegraphy (Morse) 1837;
- Electrical machinery (Sturgeon) 1832;

- Pinhole camera, 400BC, Mozi,
- Ibn Sahl, refraction 984;
- Snell, 1621;
- Huygens/Newton 1660;
- Fresnel 1814;
- Kirchhoff 1883;

- Maxwell's equations 1864/1865;
- Heaviside, Hertz, Rayleigh, Sommerfeld, Debye, Mie, Kirchhoff, Love, Lorentz (plus many unsung heroes);
- Quantum electrodynamics 1927 (Dirac, Feynman, Schwinger, Tomonaga);
- Electromagnetic technology;

- Nano-fabrication technology;
- Single-photon measurement;
- Quantum optics/Nano-optics 1980s;
- Quantum information/Bell's theorem 1980s;
- Quantum electromagnetics/optics (coming).

Figure 1.2: A brief history of electromagnetics and optics as depicted in this figure.

In Figure 1.2, a brief history of electromagnetics and optics is depicted. In the beginning, it was thought that electricity and magnetism, and optics were governed by different physical laws. Low frequency electromagnetics was governed by the understanding of fields and their interaction with media. Optical phenomena were governed by ray optics, reflection and refraction of light. But the advent of Maxwell's equations in 1865 reveal that they can be unified by electromagnetic theory. Then solving Maxwell's equations becomes a mathematical endeavor.

The photo-electric effect [20, 21], and Planck radiation law [22] point to the fact that electromagnetic energy is manifested in terms of packets of energy. Each unit of this energy is now known as the photon. A photon carries an energy packet equal to $\hbar\omega$, where ω is the angular frequency of the photon and $\hbar = 6.626 \times 10^{-34}$ J s, the Planck constant, which is a very small constant. Hence, the higher the frequency, the easier it is to detect this packet of energy, or feel the graininess of electromagnetic energy. Eventually, in 1927 [3], quantum theory was incorporated into electromagnetics, and the quantum nature of light gives rise to the field of quantum optics. Recently, even microwave photons have been measured [23]. It is a difficult measurement because of the low frequency of microwave (10^9 Hz) compared to optics (10^{15} Hz): microwave photon has a packet of energy about a million times smaller than that of optical photon.

The progress in nano-fabrication [24] allows one to make optical components that are subwavelength as the wavelength of blue light is about 450 nm. As a result, interaction of light with nano-scale optical components requires the solution of Maxwell's equations in its full glory.

In 1980s, Bell's theorem (by John Steward Bell) [25] was experimentally verified in favor of the Copenhagen school of quantum interpretation (led by Niel Bohr) [26]. This interpretation says that a quantum state is in a linear superposition of states before a measurement. But after a measurement, a quantum state collapses to the state that is measured. This implies that quantum information can be hidden in a quantum state. Hence, a quantum particle, such as a photon, its state can remain incognito until after its measurement. In other words, quantum theory is "spooky". This leads to growing interest in quantum information and quantum communication using photons. Quantum technology with the use of photons, an electromagnetic quantum particle, is a subject of growing interest.

1.3 Maxwell's Equations in Integral Form

Maxwell's equations can be presented as fundamental postulates.[5] We will present them in their integral forms, but will not belabor them until later.

$$\oint_C \mathbf{E} \cdot \boldsymbol{dl} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot \boldsymbol{dS} \qquad \text{Faraday's Law} \tag{1.3.1}$$

$$\oint_C \mathbf{H} \cdot \boldsymbol{dl} = \frac{d}{dt} \iint_S \mathbf{D} \cdot \boldsymbol{dS} + I \qquad \text{Ampere's Law} \tag{1.3.2}$$

$$\oiint_S \mathbf{D} \cdot \boldsymbol{dS} = Q \qquad \text{Gauss's or Coulomb's Law} \tag{1.3.3}$$

$$\oiint_S \mathbf{B} \cdot \boldsymbol{dS} = 0 \qquad \text{Gauss's Law} \tag{1.3.4}$$

The units of the basic quantities above are given as:

E: V/m	**H**: A/m
D: C/m^2	**B**: Webers/m^2
I: A	Q: Coulombs

[5]Postulates in physics are similar to axioms in mathematics. They are assumptions that need not be proved.

1.4 Coulomb's Law (Statics)

This law, developed in 1785 [27], expresses the force between two charges q_1 and q_2. If these charges are positive, the force is repulsive and it is given by

$$\mathbf{f}_{1\to 2} = \frac{q_1 q_2}{4\pi\varepsilon r^2}\hat{\mathbf{r}}_{12} \tag{1.4.1}$$

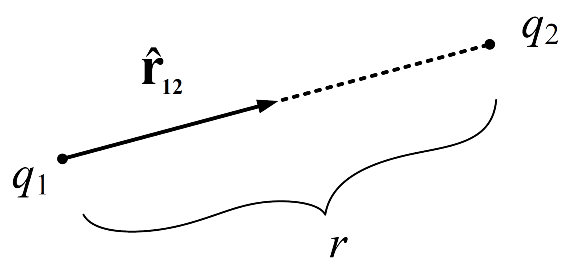

Figure 1.3: The force between two charges q_1 and q_2. The force is repulsive if the two charges have the same sign.

$\mathbf{f}$ (force): newton
q (charge): coulombs
ε (permittivity): farads/meter
r (distance between q_1 and q_2): m
$\hat{\mathbf{r}}_{12}$= unit vector pointing from charge 1 to charge 2

$$\hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}, \qquad r = |\mathbf{r}_2 - \mathbf{r}_1| \tag{1.4.2}$$

Since the unit vector can be defined in the above, the force between two charges can also be rewritten as

$$\mathbf{f}_{1\to 2} = \frac{q_1 q_2(\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon|\mathbf{r}_2 - \mathbf{r}_1|^3}, \qquad (\mathbf{r}_1, \mathbf{r}_2 \text{ are position vectors}) \tag{1.4.3}$$

1.5 Electric Field E (Statics)

The electric field **E** is defined as the force per unit charge [28]. For two charges, one of charge q and the other one of incremental charge Δq, the force between the two charges, according to Coulomb's law (1.4.1), is

$$\mathbf{f} = \frac{q\Delta q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} \tag{1.5.1}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing from charge q to the incremental charge Δq. Then the force per unit charge is given by

$$\mathbf{E} = \frac{\mathbf{f}}{\triangle q}, \qquad (\mathrm{V/m}) \tag{1.5.2}$$

This electric field **E** from a point charge q at the orgin is hence

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} \tag{1.5.3}$$

Therefore, in general, the electric field $\mathbf{E}(\mathbf{r})$ from a point charge q at $\mathbf{r}'$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|^3} \tag{1.5.4}$$

where

$$\hat{\mathbf{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{1.5.5}$$

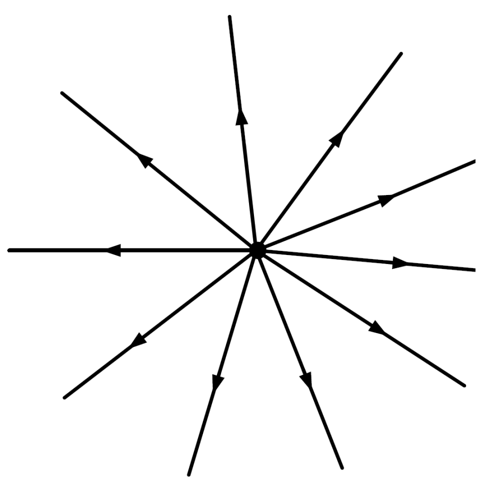

Figure 1.4: Emanating **E** field from an electric point charge as depicted by depicted by (1.5.4) and (1.5.3).

Example 1
Field of a ring of charge of density ϱ_l C/m

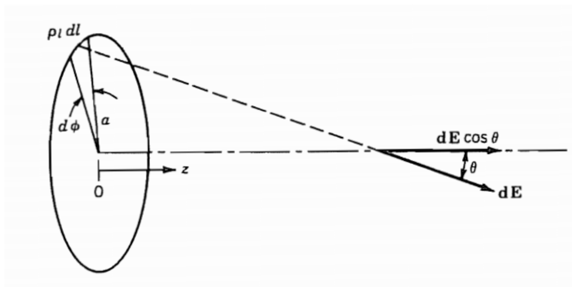

Figure 1.5: Electric field of a ring of charge (Courtesy of Ramo, Whinnery, and Van Duzer) [29].

Question: What is **E** along z axis?
Remark: If you know **E** due to a point charge, you know **E** due to any charge distribution because any charge distribution can be decomposed into sum of point charges. For instance, if there are N point charges each with amplitude q_i, then by the principle of linear superposition, the total field produced by these N charges is

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}_i|^3} \tag{1.5.6}$$

where $q_i = \varrho(\mathbf{r}_i)\Delta V_i$. In the continuum limit, one gets

$$\mathbf{E}(\mathbf{r}) = \int_V \frac{\varrho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|^3} dV \tag{1.5.7}$$

In other words, the total field, by the principle of linear superposition, is the integral summation of the contributions from the distributed charge density $\varrho(\mathbf{r})$.

1.6 Gauss's Law (Statics)

This law is also known as Coulomb's law as they are closely related to each other. Apparently, this simple law was first expressed by Joseph Louis Lagrange [30] and later, reexpressed by Gauss in 1813 (wikipedia).

This law can be expressed as

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q \tag{1.6.1}$$

$\mathbf{D}$: electric flux density C/m^2 $\mathbf{D} = \varepsilon \mathbf{E}$.

$d\mathbf{S}$: an incremental surface at the point on S given by $dS\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit normal pointing outward away from the surface.

Q: total charge enclosed by the surface S.

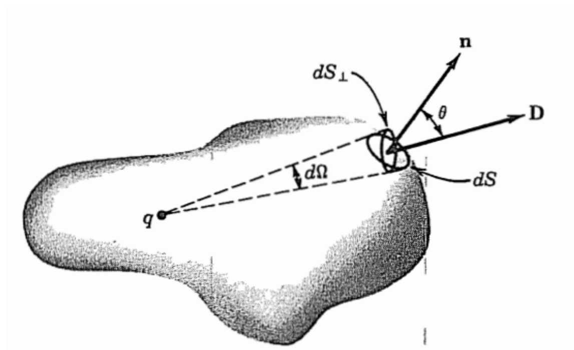

Figure 1.6: Electric flux (Courtesy of Ramo, Whinnery, and Van Duzer) [29]

The left-hand side of (1.6.1) represents a surface integral over a closed surface S. To understand it, one can break the surface into a sum of incremental surfaces ΔS_i, with a local unit normal $\hat{\mathbf{n}}_i$ associated with it. The surface integral can then be approximated by a summation

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} \approx \sum_i \mathbf{D}_i \cdot \hat{\mathbf{n}}_i \Delta S_i = \sum_i \mathbf{D}_i \cdot \Delta \mathbf{S}_i \tag{1.6.2}$$

where one has defined $\Delta \mathbf{S}_i = \hat{\mathbf{n}}_i \Delta S_i$. In the limit when ΔS_i becomes infinitesimally small, the summation becomes a surface integral.

1.7 Derivation of Gauss's Law from Coulomb's Law (Statics)

From Coulomb's law and the ensuing electric field due to a point charge, the electric flux is

$$\mathbf{D} = \varepsilon \mathbf{E} = \frac{q}{4\pi r^2} \hat{\mathbf{r}} \tag{1.7.1}$$

When a closed spherical surface S is drawn around the point charge q, by symmetry, the electric flux though every point of the surface is the same. Moreover, the normal vector $\hat{\mathbf{n}}$ on the surface is just $\hat{\mathbf{r}}$. Consequently, $\mathbf{D} \cdot \hat{\mathbf{n}} = \mathbf{D} \cdot \hat{\mathbf{r}} = q/(4\pi r^2)$, which is a constant on a spherical of radius r. Hence, we conclude that for a point charge q, and the pertinent electric flux $\mathbf{D}$ that it produces on a spherical surface,

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = 4\pi r^2 \mathbf{D} \cdot \hat{\mathbf{n}} = q \tag{1.7.2}$$

Therefore, Gauss's law is satisfied by a point charge.

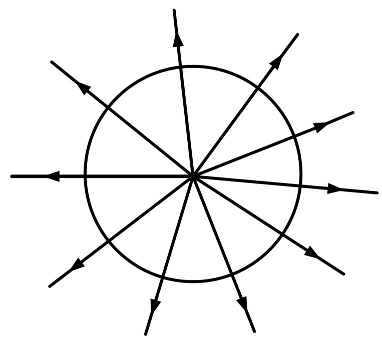

Figure 1.7: Electric flux from a point charge satisfies Gauss's law.

Even when the shape of the spherical surface S changes from a sphere to an arbitrary shape surface S, it can be shown that the total flux through S is still q. In other words, the total flux through sufaces S_1 and S_2 in Figure 1.8 are the same.

This can be appreciated by taking a sliver of the angular sector as shown in Figure 1.9. Here, ΔS_1 and ΔS_2 are two incremental surfaces intercepted by this sliver of angular sector. The amount of flux passing through this incremental surface is given by $d\mathbf{S} \cdot \mathbf{D} = \hat{\mathbf{n}} \cdot \mathbf{D} \Delta S = \hat{\mathbf{n}} \cdot \hat{\mathbf{r}} D_r \Delta S$. Here, $\mathbf{D} = \hat{\mathbf{r}} D_r$ is pointing in the $\hat{\mathbf{r}}$ direction. In ΔS_1, $\hat{n}$ is pointing in the $\hat{\mathbf{r}}$ direction. But in ΔS_2, the incremental area has been enlarged by that $\hat{\mathbf{n}}$ not aligned with $\mathbf{D}$. But this enlargement is compensated by $\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}$. Also, ΔS_2 has grown bigger, but the flux at ΔS_2 has grown weaker by the ratio of $(r_2/r_1)^2$. Finally, the two fluxes are equal in the limit that the sliver of angular sector becomes infinitesimally small. This proves the assertion that the total fluxes through S_1 and S_2 are equal. Since the total flux from a point charge q through a closed surface is independent of its shape, but always equal to q, then if we have a total charge Q which can be expressed as the sum of point charges, namely.

$$Q = \sum_i q_i \tag{1.7.3}$$

Then the total flux through a closed surface equals the total charge enclosed by it, which is the statement of Gauss's law or Coulomb's law.

Example 2

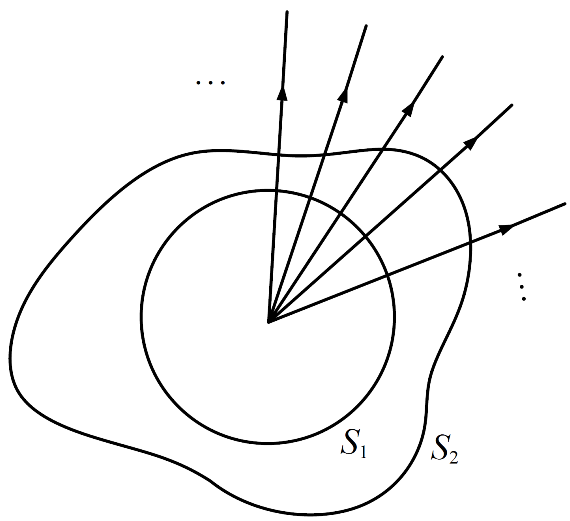

Figure 1.8: Same amount of electric flux from a point charge passes through two surfaces S_1 and S_2.

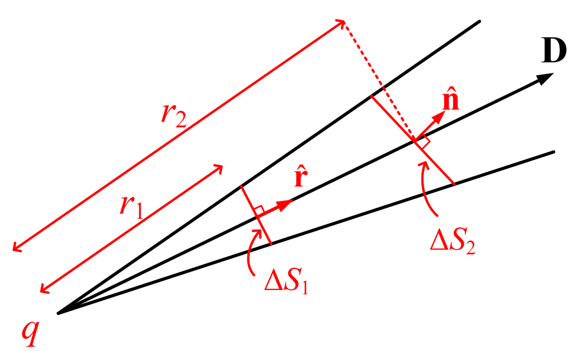

Figure 1.9: When a sliver of angular sector is taken, same amount of electric flux from a point charge passes through two incremental surfaces ΔS_1 and ΔS_2.

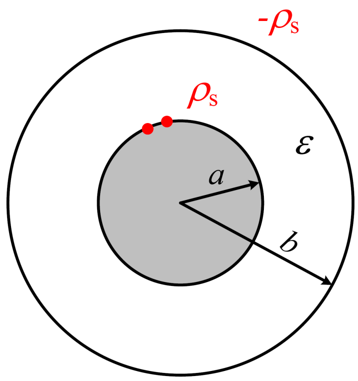

Figure 1.10: Figure for Example 2 for a coaxial cylinder.

Field between coaxial cylinders of unit length.
Question: What is $\mathbf{E}$?
Hint: Use symmetry and cylindrical coordinates to express $\mathbf{E} = \hat{\rho} E_\rho$ and appply Gauss's law.

Example 3:
Fields of a sphere of uniform charge density.

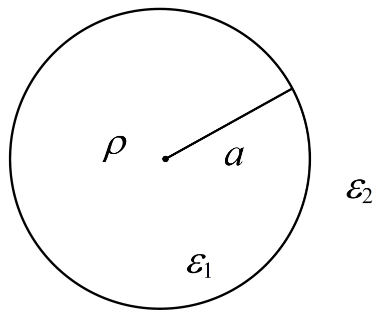

Figure 1.11: Figure for Example 3 for a sphere with uniform charge density.

Question: What is $\mathbf{E}$?
Hint: Again, use symmetry and spherical coordinates to express $\mathbf{E} = \hat{r}E_r$ and appply Gauss's law.

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